Problem Set I: due 23 January

1.   Kulsrud; Chapter 3, #1 — simple, but important educational. Highlights duality of description in terms of field lines and fluid elements.

2.   Kulsrud; Chapter 3, #3 — introduction to magnetic helicity

3.   Kulsrud; Chapter 3, #4 — simple illustration of magnetic braking. Instructive.

4.   *Electron MHD* (EMHD)

This extended problem introduces you to EMHD and challenges you to apply what you've learned about MHD to understand the structures of a different system of fluid equations. In EMHD, the ions are stationary and the "fluid" is a fluid of electrons. EMHD is useful in problems involving fast Z-pinches, filamentation and magnetic field generation in laser plasmas, Fast Igniter, etc.

The basic equations of EMHD are the electron momentum balance equation

(1) $$\frac{\partial}{\partial t}\underline{v} + \underline{v}\cdot\nabla\underline{v} = -\frac{q}{m}\underline{E} - \frac{\nabla P}{r} - \frac{q}{mc}\left(\underline{v}\times\underline{B}\right) - n\underline{v},$$

(2) $$\underline{J} = -nq\underline{v},$$

        and continuity

(3) $$\underline{\nabla}\cdot\underline{J} = 0.$$

Note that here, Ampere's law forces incompressibility of the mass flow $r\underline{v}$. Here $\underline{v}$ is the electron fluid velocity, n is the electron-ion collision frequency, $q = |e|$, $m = m_e$. Of course, Maxwell's equations apply, but the displacement current is neglected.

i.)   *Freezing-in*

Determine the freezing-in law for EMHD by taking the curl of Eqn. (1) and using the identity

$$-\underline{v} \cdot \underline{\nabla} \underline{v} = \underline{v} \times \underline{w} - \nabla\left(v^2/2\right).$$

Assume the electrons have $p = p(r)$. Approach this problem by trying to derive an equation for "something" which has the structure of the induction equation in MHD. Discuss the physics - what is the "something" and what is it frozen into? In retrospect, why is the frozen-in quantity obvious? How is freezing-in broken?

ii.)   *Large Scale Limit*

Show that for $\ell^2 \gg c^2/w_{pe}^2$, the dynamical equations for EMHD reduce to

$$\frac{\partial B}{\partial t} + \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \times \underline{B} \right) = -\eta \underline{\nabla} \times \left(\frac{\underline{J}}{nq} \right)$$

$$\underline{\nabla} \cdot \underline{J} = 0; \quad \underline{\nabla} \cdot \underline{B} = 0.$$

a)   Show that density remains constant here.

b)   Formulate an energy theorem for EMHD in this limit, by considering the energy content of a "blob" of EMHD fluid.

c)   Discuss the frozen-in law in this limit.